Laplace Transforms

Goal: Solve second order constant coefficient linear IVP's with discontinuous and impulse forcing functions by applying the Laplace transform to both sides of the equation.

For instance:

$$q''(t) + 3q'(t) + 2q(t) = H(t-3); \quad q(0) = 1, q'(0) = 0$$

$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ is called the Heaviside [1] function or sometimes the Unit Step function.

Graph $H(t-3)$. It models an "on."

What function would model "on" for one second? Give an example and graph it.

Instead of turning on one volt, how can we turn on 5 volts for 4 seconds at $t = 1$?

Definition: The **Laplace transform** of $f(t)$ is $\mathscr{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt = F(s)$ if the integral exists.

Notice if $f(t)$ can be transformed then $\lim_{t\to\infty} f(t)e^{-st} = 0$.

The Laplace transform, $\mathscr{L}$, is a linear transformation. What does that mean? Because the vectors in our vector space are functions, the linear transformation is also called a "linear operator." What are some other familiar linear operators?

[1] Oliver Heaviside, 1850 - 1925

Solve $q''(t)+3q'(t)+2q(t) = H(t-3); \quad q(0) = 1, q'(0) = 0$ using the table of Laplace transforms given on the homework.

If Θ is a linear operator so that $\Theta(y) = f(t), y(0) = 0, y'(0) = 0$ is a differential equation, then the "transfer function" is $\dfrac{1}{\mathscr{L}(\Theta)}$. What is the transfer function for the above example? A system is characterized by the transfer function.

Let's become more familiar with our table of Laplace Transforms. Prove the transform formulas for
a) $f(t) = e^{at}$ b) $f(t) = e^{ibt}$ c) $f(t) = \cos(bt)$ d) $f(t) = \sin(bt)$.

Now prove the table formula for the second derivative of a transformable function. Induction proves the nth derivative formula.

Solve $x''(t) + 9x(t) = \sin(2t); \quad x(0) = 0, x'(0) = 1.$

Prove $\mathscr{L}(t^n) = \dfrac{n!}{s^{n+1}}$. [2]

[2] If we set $s = 1$, then the transform becomes $n!$. This motivates a generalization for a factorial. The Gamma function is $\Gamma(r+1) = \int_0^\infty t^r e^{-t}\, dt$ and we can prove $\Gamma(r+1) = r\Gamma(r)$. Consequently, I might write $\Gamma(\pi+1) = \pi!$.

Prove a) $\mathscr{L}(e^{at}f(t)) = F(s-a)$, the "s-axis shift." and use it to find b) $\mathscr{L}(e^{3t}\sinh(2t))$

Prove a) $\mathscr{L}(f(t-a)H(t-a)) = e^{-as}F(s)$, the "t-axis shift." and use it to find b) $\mathscr{L}(t^2 H(t-3))$

As time allows: Find the inverse Laplace transforms, $\mathscr{L}^{-1}$, for the following.

a) $F(s) = \dfrac{s^2 + 13s - 24}{s^2(s+8)}$

b) $F(s) = \dfrac{s}{s^2 + 2s + 2}$

As time allows:
1) Prove $\mathscr{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$.

2) Prove $\mathscr{L}\left(\int_0^t f(\tau)\, d\tau\right) = \dfrac{F(s)}{s}$.