Convolution

True or False: $\mathscr{L}\left(f(t) \cdot w(t)\right) = F(s) \cdot W(s)$. Defend your answer with a proof or counterexample.

There *is* an operator for which this is true called **convolution**:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Verify the product formula for $f(t) = t$ and $g(t) = t^2$.

It is true that

$$\mathscr{L}\left(f(t) * w(t)\right) = F(s) \cdot W(s),$$

but before proving this, we will try to better understand what convolution means.

Application 1: Let $f(\tau)$ represent a "money stream." This is an investment that returns $f(\tau)$ dollars per year at time τ. The rate might vary according to time. We wish to calculate its "future value" after t years if a bank account will pay a constant interest rate of r. Draw the τ axis from 0 to t and show a small interval at random time τ of length $\Delta\tau$. Assuming $f(\tau)$ is approximately constant on this interval, about how much principal does the money stream return in that interval?

Now let $g(\tau) = e^{r\tau}$ be the "interest factor" where r is the constant rate of return. Then the "future value" of the principal over the $\Delta\tau$ interval is $f(\tau)\Delta\tau e^{r(t-\tau)}$ using the famous formula $A = Pe^{rT}$. If we sum up all of the future values from the tiny intervals we get the future value of the money stream is

$$\int_0^t f(\tau)e^{r(t-\tau)}\,d\tau$$

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= (f * g)(t).$$

Application 2 In this case $f(T)$ is the number of grams of phosphates entering a pond per day and $g(T) = e^{-rT}$ is the rate at which the phosphates decay per day. What does $(f * g)(t)$ represent in this case?

Find $\mathscr{L}\left(e^{-2t} * \sin(3t)\right)$.

Use a convolution to find $\mathscr{L}^{-1}\left(\dfrac{1}{s^2(s-1)}\right)$. Partial fractions would be just as fast if not faster.

The next problem shows convolution makes solving some "integral equations" fairly easy.

Solve $f(t) = 2t - 4\int_0^t f(t-\tau)\sin(\tau)\,d\tau$.

Properties of convolution follow. Discuss the proof of each as time allows.

(1) Prove $\mathscr{L}\left(f(t) * g(t)\right) = F(s)G(s)$.
Proof:

$$F(s)G(s) = \int_0^\infty f(\tau)e^{-s\tau}\,d\tau \cdot \int_0^\infty g(u)e^{-su}\,du$$

$$= \int_0^\infty \int_0^\infty f(\tau)g(u)e^{-s(\tau+u)}\,d\tau du$$

Let $t = \tau + u$ and $\tau = \tau$; then the Jacobian determinant is 1, so

$$F(s)G(s) = \int_0^\infty \int_0^t f(\tau)g(t-\tau)e^{-st}\,d\tau dt$$

$$= \int_0^\infty (f(t) * g(t))e^{-st}\,dt$$

$$= \mathscr{L}\left(f(t) * g(t)\right).$$

(2) Prove $f * g = g * f$.

Proof:

$$\mathscr{L}\left(f(t) * g(t)\right) = F(s)G(s) = G(s)F(s) = \mathscr{L}\left(g(t) * f(t)\right)$$

Since the Laplace transform is invertible, it must be one-to-one, hence $f * g = g * f$.

You can prove the next two easily.

(3) $(f + g) * h = f * h + g * h$

(4) $(cf) * g = c(f * g)$ if c is a constant.