# The Dirac Delta Generalized Function

An **impulse** of a force $f(t)$ over a time interval $[a, b]$ is the momentum imparted by $f(t)$ to an object over $[a, b]$. Since momentum is $mv$ and $f = \dfrac{d(mv)}{dt}$, $mv = \displaystyle\int_a^b f\,dt$ is an integral representing the impulse force of $f(t)$ over $[a, b]$. We want to model a force represented by a "kick" of a spring-mass system at time 0. We decide the "kick" instantaneously imparts one unit of momentum at time 0. We start with an estimate of $\dfrac{H(t) - H(t-h)}{h}$ where $h$ is close to 0. Draw a picture of this function.

As $h \to 0$ on our time axis we get zero everywhere which is not a good model, so we transform to $s$ space and see what happens. Find $\displaystyle\lim_{h\to 0} \mathscr{L}\left(\frac{H(t) - H(t-h)}{h}\right)$.

**Definition**: $\delta(t)$ is the generalized function for which $\mathscr{L}(\delta(t)) = 1$.

This models an impulse force, or a "kick" to the system, that imparts one unit of momentum.

Mathematicians do not like the above definition, and so they had to make new theory for these generalized functions which they call "distributions." Distributions are better for the physicist because they allow for error in any output while regular functions do not. Here is the mathematical definition: although not so intuitive as the above definition, it makes calculations with $\delta(t)$ very easy.

**Rigorous Mathematical Definition**: $\delta(t)$ is a distribution for which

$$\int_a^b \delta(t) f(t)\, dt = \begin{cases} f(0) & 0 \in [a,b] \\ 0 & 0 \notin [a,b] \end{cases}.$$

Use the rigorous definition to find $\mathscr{L}(\delta(t-a))$.

Recall the definition of a transfer function. What is the transfer function $W(s)$ and the **unit impulse response**[1] $w(t) = \mathscr{L}(W(s))$ of $y'' + 6y' + 8y = \delta(t)$ if $y(0) = 0$ and $y'(0) = 0$. The impulse response is the solution when our spring-mass-dashpot system is at rest and then kicked at time zero and imparted with one unit of momentum.

[1] Also known as the **weight function**

What is $\mathscr{L}(f(t)\delta(t-a))$?

Now solve $y'' + 6y' + 8y = f(t)\sum_{n=0}^{k}\delta(t-n)$ if $y(0) = 0$ and $y'(0) = 0$ in terms of $f$. The solution is called the **response** of the system to the forcing function $f(t)\sum_{n=0}^{k}\delta(t-n)$.

As time allows. This is just for fun.

How to find the generalized derivative of $H(t)$ at $t = 0$? Let $H_*(t) = H(t)$ for $t \neq 0$, but $H_*(0) = 0$. We feel that $H_*'(0) = H'(0)$. If we innocently use our formula for derivatives,

$$\mathscr{L}(H_*'(t)) = s\mathscr{L}(H_*(t)) - H_*(0)$$

$$= s \cdot \frac{1}{s} - 0 = 1$$

. Consequently $H'(t) = H_*'(t) = \delta(t)$.