Boundary Value Problems (BVP)

Why BVP's?

The last segment of our course covers the solution of three partial differential equations.

The one-dimensional **heat equation** (also known as the conduction equation:)

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

or

$$u_t(x,t) = k\,u_{xx}(x,t)$$

where k is a constant.

The one-dimensional **wave equation**:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = k^2\,\frac{\partial^2 y(x,t)}{\partial x^2}$$

or

$$y_{tt}(x,t) = k^2\,y_{xx}(x,t)$$

where k is a positive constant.

The two-dimensional **static heat equation** called the **Laplace Equation**:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

or

$$u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

Each equation is given **boundary conditions (B.C.'s)** and **initial conditions (I.C.'s)**.

We will use separation of variables to solve these equations; this method generates boundary value problems (BVP's) determined by the B.C.'s.

I will ask you to quickly write eigenvalues and eigenvectors for two BVP's: one from the **Dirichlet** B.C.'s and the other from the **Von Neumann** B.C.'s.

I will also ask you to find eigenvalues and eigenvectors for a variety of BVP's.

The physical system modeled by

$u_t(x,t) = k\, u_{xx}(x,t)$ with I.C. $u(x,0) = 4\sin(5\pi x)$ and B.C.'s $u(0,t) = 0$ and $u(1,t) = 0$

is a homogeneous thin metal rod with ends kept at 0° C and temperature $u(x,t)$ at position x and time t. What happens to the temperature as $t \to \infty$?

We solve this equation using **separation of variables**. The solution set is a null space of a linear operator. This method assumes we can find a basis $\{u_\lambda\}$ for the null space of the form $u_\lambda = X_\lambda(x)T_\lambda(t)$. To find such solutions we substitute $X(x)T(t)$ for u.

Now we solve the two ODE's generated from separation of variables. The equation for X has boundary values derived from the PDE's B.C.'s. This is why we study BVP's in this class. Let's finish solving the PDE pretending we can solve this BVP by inspection.

Find the eigenvalues and eigenfunctions for $X'' + \lambda X = 0$ with Dirichlet conditions: $X(0) = 0$ and $X(L) = 0$. Here $X = X(x)$. Show all steps. Repeat this problem until you have the solution memorized and know how to get it.

How do the eigenvalues and eigenfunctions change if the BC are changed to $X(a) = 0$ and $X(L+a) = 0$? Hint: use a different, translated, basis.

Find the eigenvalues and eigenfunctions for $X'' + \lambda X = 0$ with Von Neumann conditions: $X'(0) = 0$ and $X'(L) = 0$. Again, $X = X(x)$. Show all steps. Repeat this problem until you have the solution memorized and know how to get it.

Find the eigenvalues and eigenfunctions for $x'' + \lambda x = 0$ with Rabin conditions: $x(0) = 0$ and $x'(L) = 0$. In this problem, $x = x(t)$.

As time allows.

Application: A spinning string has constant tension T, linear density ρ, length L and angular velocity ω. If $y(x)$ is the height of the string at position x, then Mike Young can prove

$$Ty'' + \rho\omega^2 y = 0$$

with $y(0) = 0 = y(L)$. Notice that $y = 0$ unless $\dfrac{\rho\omega^2}{T}$ is an eigenvalue. What is the solution if $\dfrac{\rho\omega^2}{T}$ *is* an eigenvalue? Describe the behavior as ω is increased, and realize no damping is in the equation so this is an ideal model.