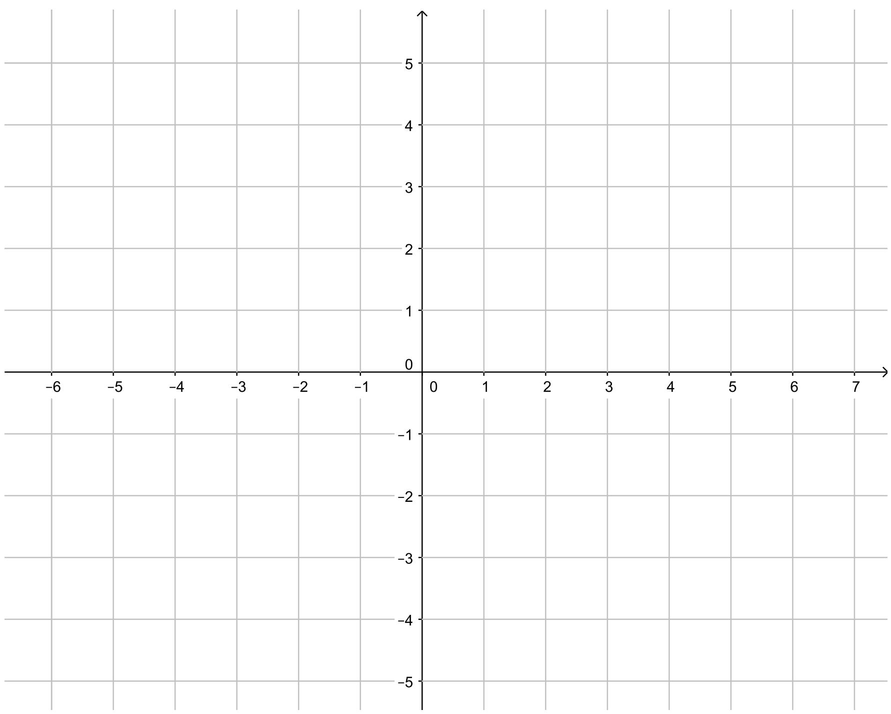
Fourier Series

$2L$-periodic functions form a vector space with inner product $< f(x), g(x) >= \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx$. The Fourier series of such a function is a projection onto the infinite dimensional subspace

$$\mathscr{F} = span\{1, \sin(\frac{n\pi}{L}x), \cos(\frac{n\pi}{L}x) | n = 1, 2, 3, ...\}$$

. In fact, $\{1, \sin(\frac{n\pi}{L}x), \cos(\frac{n\pi}{L}x) | n = 1, 2, 3, ...\}$ is an orthogonal basis for $\mathscr{F}$[1].

Let $f(x) = x$ on $(-\pi, \pi]$ and extend to a 2π-periodic function. Draw its graph. The fourier series of $f(x)$ (Hereafter denoted F.S. f) agrees with $f(x)$ wherever $f(x)$ is continuous, and is the mean of the one-sided limits of $f(x)$ when it is not continuous. Superimpose the graph of F.S. f onto the graph of $f(x)$.



Project the function $f(x)$ defined above onto $\sin(nx)$.

[1]This means $\langle 1, \sin(nx)\rangle = 0$, $\langle 1, \cos(nx)\rangle = 0$, $\langle \cos(mx), \sin(nx)\rangle = 0$, $\langle \sin(mx), \sin(nx)\rangle = 0$ if $m \neq n$, and $\langle \cos(mx), \cos(nx)\rangle = 0$ if $m \neq n$. You should prove this on your own.

What is the projection of $f(x)$ onto $\cos(nx)$, and 1?

What is the F.S. of $f(x) = x$ on $(-\pi, \pi]$ extended to a 2π-periodic function?

In general, the Fourier Series of a $2L$-periodic function $f(x)$ is

$$F.S.f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\, dx$$

.

The Fourier series is unique and it is a linear operator. Use these facts to quickly find the Fourier series for $g(x) = x + \cos^2(x)$ on $(-\pi, \pi]$ extended to a 2π-periodic function.

Let $h(x) = x^2$ on $(-1, 1]$ and extend it to a 2-periodic function. Find the Fourier Series of $h(x)$.

Solve $u_t = 2u_{xx}$ if $u_x(0,t) = 0 = u_x(1,t)$ and $u(x,0) = x^2$.

We can do this because the fourier series of $u(x,0)$ has no odd terms. What would we do if this were not the case? Extend $u(x,0)$ to an even function and then take the fourier series.