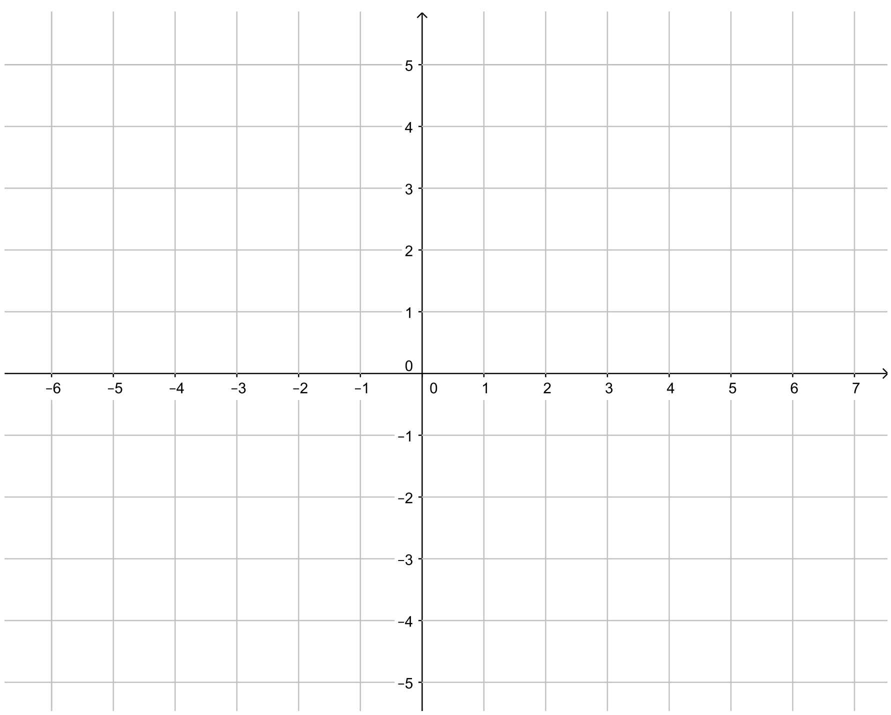
Heat Equation

Find the general solution for $u_t = 2u_{xx}$ if $u(0,t) = 0 = u(3,t)$ and $u(x,0) = 3x - x^2$.

The Fourier series of $f(x) = 3x - x^2$ on $(-3, 3)$ will have cosines as well as sines, so we can't use it to find the particular solution. Instead we find the Fourier series of the **odd extension** of $f(x)$ which I will denote as f_o. Draw the graph of f_o.



Find the F.S. f_o, called the **sine series expansion of** $f(x)$, and then write the particular solution for $u_t = 2u_{xx}$ if $u(0,t) = 0 = u(3,t)$ and $u(x,0) = 3x - x^2$.

Solve $u_t = u_{xx}$ if $u_x(0,t) = 0$, $u_x(2,t) = 0$, and $u(x,0) = H(x-1)$. Since the Neumann B.C.'s produce a cosine series for a general solution, we will have to take the even extension of $f(x) = u(x,0)$ to find the **cosine series expansion** of $f(x)$[1]. I will use $f_e(x)$ to denote the **even extension** of $f(x)$.

[1] Some B.C.'s produce a general solution with both sines and cosines. In this case we would need to use the **Fourier Extension**, where we assume $f(x)$ is periodic of period 2, not 4.

Solve $u_t = u_{xx}$ if $u(0,t) = 10$, $u(2,t) = 0$, and $u(x,0) = 3\sin(5\pi x) - 5x + 10$. Use superposition.

How do we derive the heat equation? It's best done for the 3-D version; we can obtain the 1-D version by setting y and z equal to zero. Let $u(x, y, z, t)$ equal the temperature of some homogeneous material at a point (x, y, z) at time t. Also let

$H(t)$ = Heat content of a solid chunk in region E at time t = $\iiint_E k_1 u(x, y, z, t)\, dV$ where k_1 is a constant depending on the properties of the material.

We express $H'(t)$ in two different ways and set them equal.

$$(1) \qquad H'(t) = \frac{d\left(\iiint_E k_1 u(x, y, z, t)\, dV\right)}{dt} = \iiint_E k_1 \frac{\partial u(x, y, z, t)}{\partial t}\, dV$$

But $H'(t)$ is also proportional to the flux of ∇u **out** of the boundary of E, since if it is hotter outside E then the temperature of E will increase, so

$$H'(t) = \oiint_{Bd(E)} k_2 \nabla u \cdot d\vec{S}$$

and then applying the divergence theorem gives us:

$$(2) \qquad H'(t) = \iiint_E k_2 \nabla \cdot \nabla u\, dV.$$

Then (1) = (2) gives

$$\iiint_E k_2 \nabla \cdot \nabla u\, dV = \iiint_E k_1 \frac{\partial u(x, y, z, t)}{\partial t}\, dV$$

for every region E, no matter how small. If we assume u is continuously differentiable, then the integrands must be equal. Setting $k = \frac{k_2}{k_1}$ gives us the 3-D heat equation:

$$u_t = k(u_{xx} + u_{yy} + u_{zz}).$$

If y and z are identically zero, we then have the 1-D equation.

As time allows: how can we find temperature on a ring with insulated sides. We will use the 1-D heat equation over an interval $[-L, L]$ where the ring meets at $x = L$ and $x = -L$: so $u(L, t) = u(-L, t)$ and $u_x(L, t) = u_x(-L, t)$ are the boundary conditions. Let $u(x, 0) = f(x)$ be the initial condition.