Quiz 5 25 points Math 220 Name: ____________________

1. (6 points) Compute e^{At} if $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. Without much more work, write the solution for $\frac{d\vec{u}}{dt} = A\vec{u}$, $\vec{u}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ as a single simplified vector.

$$\begin{matrix} \tau = 6 \\ \Delta = 9 \end{matrix} \Rightarrow \lambda = 3, 3. \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow N(A-3I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\therefore e^{At} = e^{3It} e^{(A-3I)t} = e^{3t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t\right)$$

$$\Rightarrow \boxed{e^{At} = e^{3t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}}.$$

Since $e^{A0} = e^{[0]} = I$, $\vec{u}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\Rightarrow \vec{u} = e^{At}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \boxed{e^{3t}\begin{bmatrix} 2-3t \\ 5-3t \end{bmatrix}}$$

2. (6 points) Find a **particular** solution for $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t)$ if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and if $\vec{f}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}$.

$$\begin{matrix} \tau = 2 \\ \Delta = 0 \end{matrix} \Rightarrow \lambda = 0, 2.$$ A symmetric, so spectral theorem and $N(A) = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\Rightarrow N(A-2I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \therefore W = \begin{vmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{vmatrix} = 2e^{2t}.$$

$$V_1' = \frac{\begin{vmatrix} 0 & e^{2t} \\ t & e^{2t} \end{vmatrix}}{2e^{2t}} = -\frac{t}{2} \Rightarrow V_1 = -\frac{t^2}{4}.$$

$$V_2' = \frac{\begin{vmatrix} 1 & 0 \\ -1 & t \end{vmatrix}}{2e^{2t}} = \frac{t}{2}e^{-2t} \Rightarrow V_2 = -\frac{t}{4}e^{-2t} - \frac{e^{-2t}}{8}$$

Scratch

+	$\frac{t}{2}$	e^{-2t}
−	$\frac{1}{2}$	$-\frac{e^{-2t}}{2}$
	0	$\frac{e^{-2t}}{4}$

$$\therefore \vec{u}_p = -\frac{t^2}{4}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-2t}\left(\frac{t}{4} + \frac{1}{8}\right)e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{u}_p = \frac{-1}{8}\begin{bmatrix} 2t^2 + 2t + 1 \\ -t^2 + 2t + 1 \end{bmatrix}}$$

Q5

3. (7 points) Find and classify each critical point for $\begin{aligned} x' &= xy + y \\ y' &= x + y - x^2 \end{aligned}$

$xy + y = 0 \Rightarrow y = 0$ or $x = -1$

$y = 0$ and $y' = 0 \Rightarrow x - x^2 = 0 \Rightarrow x = 0, 1$

$x = -1$ and $y' = 0 \Rightarrow y = 2$

$\Rightarrow$ c.p.'s are $(1,0), (0,0), (-1,2)$

$J(x,y) = \begin{bmatrix} y & x+1 \\ 1-2x & 1 \end{bmatrix}$

$J(1,0) = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$ $\begin{matrix} \tau = 1 \\ \Delta = 2 \end{matrix} \Rightarrow \tau^2 - 4\Delta < 0$

$\Rightarrow$ $\boxed{(1,0) \text{ is an unstable spiral}}$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \Delta = -1 < 0 \Rightarrow$ $\boxed{(0,0) \text{ is an unstable saddle}}$

$J(-1,2) = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow \lambda = 2, 1$

$\Rightarrow$ $\boxed{(-1,2) \text{ is an unstable node}}$

4. (6 points) Solve $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t)$ if $A = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ and if $\vec{f}(t) = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} = e^{-t}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$\begin{matrix} \tau = -2 \\ \Delta = 5 \end{matrix} \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$\left[\begin{array}{cc|c} -2i & -1 & 0 \\ \cancel{4} & \cancel{2i} & 0 \end{array}\right] \Rightarrow N(A - \lambda_+ I) = \varepsilon\begin{bmatrix} -1 \\ 2i \end{bmatrix} \Rightarrow \vec{u}_c = e^{-t}\left(c_1 \begin{bmatrix} -\cos(2t) \\ -2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(2t) \\ 2\cos(2t) \end{bmatrix}\right)$

or $\boxed{\vec{u}_c = e^{-t}\left(c_3 \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(2t) \\ 2\cos(2t) \end{bmatrix}\right)}$

Guess:

$\vec{u}_p = e^{-t}\vec{K}$, $\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ constant.

$\Rightarrow -\cancel{e^{-t}}\vec{K} = A\cancel{e^{-t}}\vec{K} + \cancel{e^{-t}}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{K} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & -1 & -2 \\ 4 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} K_2 = 2 \\ K_1 = 0 \end{matrix} \Rightarrow \boxed{\vec{u}_p = e^{-t}\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$

and $\boxed{\vec{u} = \vec{u}_c + u_p}$