Quiz 5 25 points Math 220 Name: ______________________

1. (6 points) Compute e^{At} if $A = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix}$. Without ~~any further~~ much more work, write the solution for $\frac{d\vec{u}}{dt} = A\vec{u}$, $\vec{u}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

$\begin{matrix} \tau = -4 \\ \Delta = 4 \end{matrix} \Rightarrow \lambda = -2, -2. \quad \therefore e^{At} = e^{-2t}\left[I + \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} t \right]$

$\Rightarrow \boxed{e^{At} = e^{-2t}\begin{bmatrix} 1+3t & -3t \\ 3t & 1-3t \end{bmatrix}}.$ and a solution to $\vec{u}' = A\vec{u}$, $\vec{u}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

is $\vec{u}(t) = e^{-2t}\begin{bmatrix} 1+3t & -3t \\ 3t & 1-3t \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (notice $e^{At} = I$ if $t=0$)

$\Rightarrow \boxed{\vec{u}(t) = e^{-2t}\begin{bmatrix} 2-9t \\ 5-9t \end{bmatrix}}$ or $e^{-2t}\left(2\begin{bmatrix} 1+3t \\ 3t \end{bmatrix} + 5\begin{bmatrix} -3t \\ 1-3t \end{bmatrix} \right)$

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2. (6 points) Solve $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t)$ if $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ and if $\vec{f}(t) = e^{3t}\begin{bmatrix} -8 \\ 4 \end{bmatrix}$.

$\begin{matrix} \tau = 4 \\ \Delta = 4 \end{matrix} \Rightarrow \lambda = 2, 2; \quad \left[A - 2I \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] = \left[\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] \Rightarrow N(A-2I) = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ defect.

$\left[\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ -1 \end{matrix} \right] \Rightarrow \vec{v}^* \in c\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \boxed{\vec{u}_c = e^{2t}\left[c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t \right) \right]}.$

$\vec{u}_p = \vec{K}e^{3t} \Rightarrow 3\vec{K}\cancel{e^{3t}} = \cancel{e^{3t}}A\vec{K} + \cancel{e^{3t}}\begin{bmatrix} -8 \\ 4 \end{bmatrix}$

$\Rightarrow (A - 3I)\vec{K} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \Rightarrow \left[\begin{matrix} 0 & 1 \\ -1 & -2 \end{matrix} \,\middle|\, \begin{matrix} 8 \\ -4 \end{matrix} \right] \Rightarrow \vec{K} = \begin{bmatrix} -12 \\ 8 \end{bmatrix}.$

$\boxed{\therefore \vec{u} = \vec{u}_c + e^{3t}\begin{bmatrix} -12 \\ 8 \end{bmatrix}}$

3. (7 points) Find and classify each critical point for $\begin{aligned} x' &= x^2 - y^2 \\ y' &= x^2 + y^2 - 8 \end{aligned}$

$x^2 - y^2 = 0 \Rightarrow \underline{x^2 = y^2}$

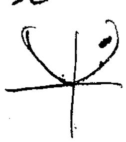
$x^2 + y^2 - 8 = 0 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2 \Rightarrow y = \pm 2$, so there are $\underline{\text{four}}$ c.p.'s, $(\pm 2, \pm 2)$.

$J(x,y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$

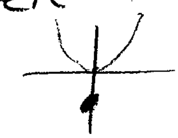
$\Rightarrow J(2,2) = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$

$\tau = 8,\ \Delta = 32$

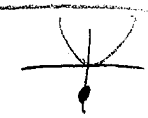
$\tau^2 - 4\Delta < 0$



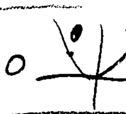
$J(2,-2) = \begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$; $\tau = 0$, $\Delta = -32$



$J(-2,2) = \begin{bmatrix} -4 & -4 \\ -4 & 4 \end{bmatrix}$ $\tau = 0$, $\Delta = -32$



$J(-2,-2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$ $\tau = -8$, $\Delta = 32$, $\tau^2 - 4\Delta < 0$



$\therefore$ $\underline{(2,-2) \text{ and } (-2,2) \text{ are unstable saddles}}$

$\underline{(2,2) \text{ is an unstable spiral}}$

and $\underline{(-2,-2) \text{ is a stable spiral.}}$

4. (6 points) Find a **particular** solution for $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t)$ if $A = \begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix}$ and if $\vec{f}(t) = \begin{bmatrix} 0 \\ \sec(2t) \end{bmatrix}$.

$\begin{aligned}\tau &= 0 \\ \Delta &= 4\end{aligned} \Rightarrow \lambda = \pm 2i.\ \left[A - 2iI \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix} 1-2i & -1 \\ 5 & -1-2i \end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] \Rightarrow c\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}.$

$\vec{u}_c = c_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}}_{\vec{X}_1} + c_2 \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}}_{\vec{X}_2}$ $\qquad \vec{u}_p = V_1\vec{X}_1 + V_2\vec{X}_2.$

$\therefore\ \Phi = \begin{bmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) + 2\sin(2t) & \sin(2t) - 2\cos(2t) \end{bmatrix} \Rightarrow W = -2\cos^2(2t) - 2\sin^2(2t) = -2.$

$V_1' = \dfrac{\begin{vmatrix} 0 & \sin(2t) \\ \sec(2t) & * \end{vmatrix}}{-2} = +\frac{1}{2}\tan(2t) \Rightarrow \underline{V_1 = -\frac{1}{4}\ln|\cos(2t)|}.$

$V_2' = \dfrac{\begin{vmatrix} \cos(2t) & 0 \\ & \sec(2t) \end{vmatrix}}{-2} = -\frac{1}{2} \Rightarrow \underline{V_2 = -\frac{t}{2}}.$

$\therefore\ \boxed{\vec{u}_p = \left(-\frac{1}{4}\ln|\cos(2t)|\right)\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} - \frac{t}{2}\begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}}$