Quiz 6 25 points Math 220 Name: ____________________

1. (6 points) Solve $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$ if $y(1) = 2$ and $y'(1) = 4$.

$y = x^r$

$\Rightarrow 4r^2 + 4r + 1 = 0$

$\Rightarrow (2r+1)^2 = 0$

$\Rightarrow r = -\frac{1}{2}, -\frac{1}{2}$

$\Rightarrow y = c_1 x^{-\frac{1}{2}} + c_2 \ln(x) x^{-\frac{1}{2}}$

$y(1) = 2 \Rightarrow 2 = c_1$

$y'(1) = 4 \Rightarrow -\frac{1}{2}\cdot 2 + 0 + c_2 = 4$

$\Rightarrow c_2 = 5.$

$\boxed{\therefore y = 2x^{-\frac{1}{2}} + 5\ln(x)\, x^{-\frac{1}{2}}}$

2. Without doing any work, write the general solution for the following equations. without much work.

(a) (2 points) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (3x^2 - 2)y = 0.$

$\boxed{y = c_1 J_{\sqrt{2}}(\sqrt{3}x) + c_2 Y_{\sqrt{2}}(\sqrt{3}x)}$

(b) (2 points) $x\frac{d\left[x\frac{dy}{dx}\right]}{dx} - 25y + 9x^2y = 0.$

$x(xy'' + y') + (9x^2 - 25)y = 0 \Rightarrow x^2y'' + xy' + (9x^2 - 25)y = 0$

$\Rightarrow \boxed{y = c_1 J_5(3x) + c_2 Y_5(3x)}$

3. (3 points) Find the singular points for $x^3(x-1)^2\frac{d^2y}{dx^2} + x(x-1)\frac{dy}{dx} + xy = 0$, and then classify them as regular or irregular.

$\boxed{\text{Singular points: } x = 0, 1}$

$x \cdot \frac{x(x-1)}{x^3(x-1)^2} = \frac{1}{x(x-1)} \Rightarrow x = 0$ is an irregular singular point.

$\left.\begin{array}{l} (x-1)\cdot \frac{x(x-1)}{x^3(x-1)^2} = \frac{1}{x^2} \\ (x-1)^2 \cdot \frac{x}{x^3(x-1)^2} = \frac{1}{x^2} \end{array}\right\} \Rightarrow x = 1$ is a regular singular point.

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4. (12 points) Find one solution to $x^2\frac{d^2y}{dx^2} + (x^3+x)\frac{dy}{dx} - y = 0$. Use the first three nonzero terms of a series solutions.

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad \underline{c_0 \neq 0}.$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n (n+r) x^{n+r+2}}_{k=n+2} + \underbrace{\sum_{n=0}^{\infty} c_n (n+r) x^{n+r}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} (-1) c_n x^{n+r}}_{k=n} = 0$$

$$\underset{\div x^r}{\Rightarrow} \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^k + \sum_{k=2}^{\infty} c_{k-2} (k+r-2) x^k + \sum_{k=0}^{\infty} c_k (k+r) x^k + \sum_{k=0}^{\infty} (-1) c_k x^k = 0$$

$$\underline{k=0} \Rightarrow c_0 r(r-1) + c_0 r - c_0 = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow \boxed{r = \pm 1.}$$

$$k=1 \Rightarrow c_1 (r+1) r + c_1 (r+1) - c_1 = 0 \Rightarrow \boxed{c_1 = 0} \text{ or } r^2 + 2r = 0$$

$$\Rightarrow \boxed{\cancel{r=0} \text{ or } \cancel{r=-2}}$$

not equal so cannot be true.

$$k \geq 2 \Rightarrow c_k = \frac{-c_{k-2}(k+r-2)}{(k+r)(k+r-1) + [(k+r) - 1]}$$

$$\Rightarrow c_k = \frac{-c_{k-2}(k+r-2)}{(k+r-1)[k+r+1]}, \quad k \geq 2. \qquad \underline{r=-1} \text{ and } k=2 \Rightarrow c_2 = \frac{-c_0(-1)}{(0)(2)}$$

impossible

$$\therefore \underline{r=1}: \quad \boxed{c_k = \frac{-c_{k-2}(k-1)}{k(k+2)}, \quad k \geq 2}$$

$$\left.\begin{array}{l} c_0 = 1 \\ c_1 = 0 \end{array}\right\} \Rightarrow c_2 = \frac{-1}{8} \Rightarrow c_3 = 0 \Rightarrow c_4 = \frac{\frac{1}{8}(3)}{4(6)} = \frac{1}{64}.$$

$\therefore$ a solution is

$$\boxed{y = x\left[1 - \frac{x^2}{8} + \frac{x^4}{64} - \cdots\right]}$$