Quiz 7 25 points Math 220 Name: ______________________

1. (7 points) Solve $y''(t) + 4y(t) = tH(t-2) + 2t\delta(t-1)$ if $y(0) = 0$ and $y'(0) = 0$.

$\xrightarrow{\mathcal{L}}$ $s^2 Y + 4Y = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + 2e^{-s}$ $\leftarrow$

$f(t-2) = t$

$\Rightarrow f(t) = t+2$

$\Rightarrow F(s) = \frac{1}{s^2} + \frac{2}{s}$

$\Rightarrow$ $Y = e^{-2s}\left(\frac{2s+1}{s^2(s^2+4)}\right) + 2e^{-s}\left(\frac{1}{s^2+4}\right)$

$\Rightarrow$ $Y = e^{-2s}\left(\frac{\frac{1}{4}}{s^2} + \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s + \frac{1}{4}}{s^2+4}\right) + e^{-s}\left(\frac{2}{s^2+4}\right)$ $\leftarrow$

$\frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+4}$

$B = \frac{d}{ds}\frac{2s+1}{s^2+4}\Big|_{s=0} = \frac{8-0}{16}$

$C(2i)+D = \frac{4i+1}{-4} = -i - \frac{1}{4}$

$C = -\frac{1}{2},\ D = -\frac{1}{4}$

$\Rightarrow$ $\boxed{y(t) = H(t-2)\left[\frac{1}{4}(t-2) + \frac{1}{2} - \frac{1}{2}\cos(2(t-2)) - \frac{1}{8}\sin(2(t-2))\right] + H(t-1)\sin(2(t-1))}$

OR $\boxed{y(t) = H(t-2)\left[\frac{1}{4}t - \frac{1}{2}\cos(2(t-2)) - \frac{1}{8}\sin(2(t-2))\right] + H(t-1)\sin(2(t-1))}$

$\downarrow \downarrow$ space

2. (6 points) Solve $f(t) = t - \underbrace{\int_0^t (t-\tau)f(\tau)d\tau}_{t * f}$.

$\xrightarrow{\mathcal{L}}$ $F(s) = \frac{1}{s^2} - \frac{1}{s^2}F(s)$

$\Rightarrow$ $F(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^2}$

$\Rightarrow$ $F(s) = \frac{1}{s^2} \cdot \frac{1}{1 + \frac{1}{s^2}} = \frac{1}{s^2+1}$

$\Rightarrow$ $\boxed{f(t) = \sin(t)}$

$\downarrow \downarrow$

Q7

3. (6 points) Use the definition of Laplace transform to find $\mathcal{L}\{t^2\}$.

$$\mathcal{L}(t^2) = \int_0^{\infty} t^2 e^{-st}\, dt$$

+	t^2	e^{-st}
−	$2t$	$\frac{e^{-st}}{-s}$
+	2	$\frac{e^{-st}}{s^2}$
	0	$\frac{e^{-st}}{-s^3}$

$$= \left(-\frac{t^2 e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right)\Bigg|_0^{\infty}$$

$$= (0 - 0 - 0) - \left(0 - 0 - \frac{2}{s^3}\right)$$

$$= \frac{2}{s^3} \quad \blacksquare$$

4. (6 points) Find the Laplace transform of $f(t) = (t^2 - 4t + 2)\delta(t-1) + \cos(t)H(t-\pi)$. Show work.

$$\mathcal{L}(f(t)) = \underbrace{\int_0^{\infty} (t^2 - 4t + 2)\,\delta(t-1)\, e^{-st}\, dt}_{} + e^{-\pi s}\mathcal{L}(\cos(t+\pi))$$

To evaluate, use def^n of $\delta(t-1)$ & sub 1 in for t in rest of integrand.

$g(t-\pi) = \cos(t)$

$\Rightarrow g(t) = \cos(t+\pi)$

$= \cos t \cos(\pi) - \sin(t)\sin(\pi)$

$= -\cos(t)$

$$= (1 - 4 + 2)e^{-s} + e^{-\pi s}\mathcal{L}(-\cos(t))$$

$$= \boxed{-e^{-s} - e^{-\pi s} \cdot \frac{s}{s^2+1}}$$