Quiz 8 25 points Math 220 Name: ____________________

1. (7 points) Find the eigenvalues and eigenfunctions for $x''(t) + \lambda x(t) = 0$, $x'(1) = 0$, and $x(5) = 0$. Show detailed work.

$$x(t) = \begin{cases} A\cos(\sqrt{\lambda}(t-1)) + B\sin(\sqrt{\lambda}(t-1)), & \lambda > 0 \\ A + B(t-1), & \lambda = 0 \\ A\cosh(\sqrt{-\lambda}(t-1)) + B\sinh(\sqrt{-\lambda}(t-1)), & \lambda < 0 \end{cases}$$

Note: $\cos\left(\frac{(2n-1)\pi}{8}(t-1)\right) = \sin\left(\frac{(2n-1)\pi}{8}(t-5)\right)$, so it doesn't matter which endpoint you start with. Proof: use $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

$0 = x'(1) \Rightarrow B = 0 \ \forall \lambda.$

$$0 = x(5) \Rightarrow \begin{cases} A\cos(4\sqrt{\lambda}) = 0, & \lambda > 0 \\ A = 0, & \lambda = 0 \\ A\sqrt{-\lambda}\sinh(4\sqrt{-\lambda}) = 0, & \lambda < 0 \end{cases} \Rightarrow \begin{cases} 4\sqrt{\lambda} = (2n-1)\frac{\pi}{2}, \ n = 1, 2, \ldots \\ A = 0 \ \forall \lambda \le 0 \end{cases}$$

$\therefore$ the evals are $\lambda = \frac{(2n-1)^2\pi^2}{64}$, $n = 1, 2, \ldots$ and the espaces have basis $x_n = \cos\left(\frac{(2n-1)\pi}{8}(t-1)\right)$

2. (6 points) Solve $u_t = 5u_{xx}$ if $u(0,t) = 0$, $u(2,t) = 0$, and $u(x,0) = -4\sin(3\pi x) + \sin(6\pi x)$. Show the steps discussed in class.

$u = XT \Rightarrow T' = -5\lambda T$ and $X'' + \lambda X = 0$, $X(0) = 0$, $X(2) = 0$

$\Rightarrow T_\lambda = e^{-5\lambda t}$ and $\lambda = \frac{n^2\pi^2}{4}$, $X_\lambda = \sin(\sqrt{\lambda}x)$, $n = 1, 2, \ldots$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{5n^2\pi^2 t}{4}} \sin\left(\frac{n\pi}{2}x\right).$$

$$-4\sin(3\pi x) + \sin(6\pi x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right)$$

$\Rightarrow c_6 = -4$, $c_{12} = 1$, other $c_n = 0$.

$$\therefore u(x,t) = -4e^{-\frac{5(6)^2\pi^2 t}{4}}\sin(3\pi x) + e^{-\frac{5(12)^2\pi^2 t}{4}}\sin(6\pi x)$$

$$= -4e^{-45\pi^2 t}\sin(3\pi x) + e^{-180\pi^2 t}\sin(6\pi x)$$

3. (6 points) Let $f(x) = 1 - x$ for $0 \le x \le 1$. Find the ~~sine and~~ cosine series expansions for the even and odd extensions of $f(x)$.

F.S. f_E: $a_0 = \frac{1}{1}\int_{-1}^{1} f_E\,dx = 2\int_0^1 1-x\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^1 = 1.$

$b_n = 0$

$a_n = \frac{1}{1}\int_{-1}^{1} f_E \cos(n\pi x)\,dx = 2\int_0^1 (1-x)\cos(n\pi x)\,dx$

$$\begin{array}{c|c} + \; 1-x & \cos(n\pi x) \\ - \; -1 & \frac{1}{n\pi}\sin(n\pi x) \\ \hline 0 & \frac{-1}{n^2\pi^2}\cos(n\pi x) \end{array}$$

$= 2\left[\cancel{\frac{1-x}{n\pi}\sin(n\pi x)} - \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^1\right.$

$= \frac{2}{n^2\pi^2}\left(1 - (-1)^n\right) = \begin{cases} \frac{4}{n^2\pi^2}, & n \text{ odd} = 1, 3, 5, \ldots \\ 0, & n \text{ even} \end{cases}$

$\therefore$ $\boxed{\text{Cosine Series} = F.S. f_E = \frac{1}{2} + \sum_{n\ \text{odd}}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi x)}$

4. (6 points) Solve $u_t = 5u_{xx}$ if $u_x(0,t) = 0$, $u_x(1,t) = 0$, and $u(x,0) = 1 - x$. Show the steps discussed in class but use work from earlier questions. ~~Use your answer to #3~~.

$u = XT \Rightarrow T' = -5\lambda T$ and $X'' + \lambda X = 0$, $X'(0) = 0 = X'(1)$.

$\Rightarrow T_\lambda = e^{-5\lambda t}$; $\lambda = n^2\pi^2$; $n = 0, 1, 2, \ldots$; $X_n = \cos(n\pi x)$

$\Rightarrow u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-5n^2\pi^2 t}\cos(n\pi x).$

$\frac{1}{2} + \sum_{n\ \text{odd}}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi x) = 1 - x = u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n\cos(n\pi x)$

$\Rightarrow \boxed{u(x,t) = \frac{1}{2} + \sum_{n\ \text{odd}}^{\infty} \frac{4}{n^2\pi^2} e^{-5n^2\pi^2 t}\cos(n\pi x)}$