Test 2 100 points Math 220 Name: ______________________

1. 4 (~~6~~ points) ~~(a)~~ Write $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 6y = t$ as an autonomous system of first order ~~differential~~ equations by introducing a new ~~in~~dependent variable. Write your final answer in $\vec{u}' = A\vec{u}$ form. $+f(t)$

$+f(t)$

~~(b) What is the critical point?~~

$$\begin{bmatrix} y \\ y' \\ t \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -8 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

~~In~~dependent

2. 12 (~~10~~ points) (a) Solve $\begin{aligned} x' &= x + 2y - 1 \\ y' &= 2x + y + 4 \end{aligned}$. (b) What is the critical point?

(c) Draw the phase diagram for the system on the provided axes.

a) $\tau = 2$, $\Delta = -3 \Rightarrow \lambda = 3, -1$.

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow N(A-3I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow N(A+I) = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{u}_c = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\vec{u}_p = \vec{K} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\vec{K} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -6 \end{array}\right]$$

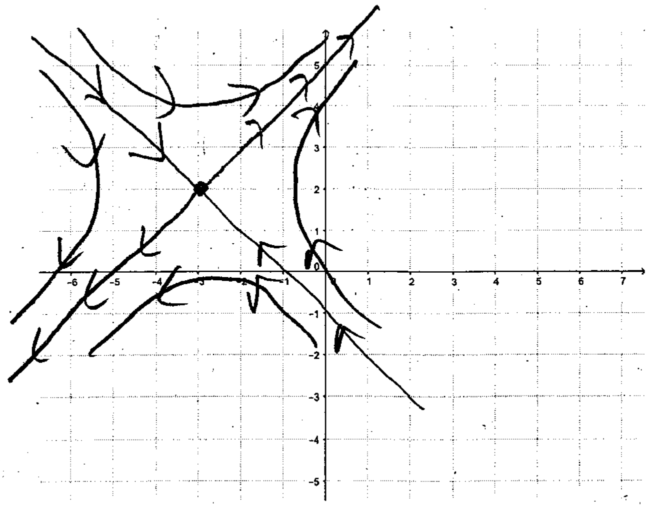
$$\Rightarrow \vec{K} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

$$\boxed{\therefore \vec{u}_p = \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$ and

$$\boxed{\vec{u} = \vec{u}_c + \vec{u}_p}$$

b) $\begin{aligned} x + 2y - 1 &= 0 \\ 2x + y + 4 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & -4 \end{array}\right]$ as in (a),

$$\boxed{\text{So the c.p. is } \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$



3. (9 points) Find and classify the critical points for $\begin{array}{l} x' = (3-x)(y-x) \\ y' = (x-2)(y-1) \end{array}$.

$X' = 0 \Rightarrow \underline{x = 3}$ or $\underline{x = y}$

$\underline{y' = 0 \text{ and } x = 3} \Rightarrow 1(y-1) = 0 \Rightarrow \underline{y = 1}$

$\underline{y' = 0 \text{ and } x = y} \Rightarrow x = 2$ or $y = 1$

$\Rightarrow \boxed{\text{c.p.'s are } (3,1), (2,2), (1,1)}$

$$J(x,y) = \begin{bmatrix} x-y+x-3 & 3-x \\ y-1 & x-2 \end{bmatrix}$$

$$J(3,1) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$\underline{\text{unstable node}}$, $\lambda = 2, 1$.

$$J(2,2) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{array}{l} \tau = -1 \\ \Delta = -1 < 0 \Rightarrow \text{Saddle} \end{array}$$

$$J(1,1) = \begin{bmatrix} -2 & 2 \\ 0 & -1 \end{bmatrix} \text{ stable node, } \lambda = -2, -1.$$

$\therefore$ (3,1) is an unstable node
(2,2) is an unstable saddle
(1,1) is a stable node

4. (8 points) Solve $4x^2\dfrac{d^2y}{dx^2} + 16x\dfrac{dy}{dx} + 9y = 0$ if $y(1) = 0$ and $y'(1) = 1$.

$4r^2 + 12r + 9 = 0$

$(2r+3)^2 = 0$

$r = -\frac{3}{2}, -\frac{3}{2}$

$y = C_1 x^{-\frac{3}{2}} + C_2 \ln(x) x^{-\frac{3}{2}}$.

$y(1) = 0 \Rightarrow \underline{0 = C_1}$.

$y'(1) = 1 \Rightarrow 1 = C_2\left[x^{-\frac{5}{2}} - \frac{3}{2}\ln(x) x^{-\frac{5}{2}}\right]\Big|_{x=1}$

$\Rightarrow 1 = C_2 [1 - 0]$

$\therefore$

$$\boxed{y = \ln(x) x^{-\frac{3}{2}}}$$

* do the spirals

5. (~~10~~ 11 points) (a) Solve the IVP $\vec{u}\,' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \vec{u}$ if $\vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) The critical point for the system is a spiral; does it move clockwise or counterclockwise? Show work to defend your answer.

a) $\tau = 2$, $\Delta = 5 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \underline{1 \pm 2i}$

$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \Rightarrow N(A-(1+2i)I) = c\begin{bmatrix} 1 \\ i \end{bmatrix}$. $e^{2it}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + i\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

Then $\vec{u}(t) = e^{t}\left(c_1\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}\right)$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow c_1 = 1,\ c_2 = 0.$

$$\boxed{\therefore\ u(t) = e^{t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}$$

b) $\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$; $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 < 0 \Rightarrow$ **Clockwise**

6. Without doing any work, write the general solution for the following equations.

~~(a) (2 points) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 0.01)y = 0.$~~ ✓

(b) (3 points) $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$ if the roots to the characteristic equation are $3 \pm 2i$.

$$\boxed{y = x^3\left(c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right)}$$

(c) (~~2~~ 3 points) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (4x^2 - 1)\,y = 0.$

$$\boxed{y = c_1 J_1(2x) + c_2 Y_1(2x)}$$

7. (10 points) Find a **particular** solution for $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t)$ if $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and if $\vec{f}(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}$.

$\begin{matrix} \tau = 0 \\ \Delta = -1 \end{matrix} \Rightarrow \lambda = \pm 1$; $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow N(A-I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \Rightarrow N(A+I) = c\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$\therefore W = \begin{vmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{vmatrix} = 3 - 1 = 2$$

$$v_1' = \frac{\begin{vmatrix} e^t & e^{-t} \\ 1 & 3e^{-t} \end{vmatrix}}{2} = \frac{3 - e^{-t}}{2} \Rightarrow v_1 = \frac{1}{2}(3t + e^{-t})$$

$$v_2' = \frac{\begin{vmatrix} e^t & e^t \\ e^t & 1 \end{vmatrix}}{2} = \frac{e^t - e^{2t}}{2} \Rightarrow v_2 = \frac{1}{2}\left(e^t - \frac{e^{2t}}{2}\right)$$

$$\therefore \boxed{\vec{u}_p = \frac{1}{2}(3t + e^{-t})\begin{bmatrix} e^t \\ e^t \end{bmatrix} + \frac{1}{2}\left(e^t - \frac{e^{2t}}{2}\right)\begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}}$$

$$\text{or} \quad \frac{1}{2}(3te^t + 1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(1 - \frac{e^t}{2}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{e^t}{4} + \frac{3te^t}{2} \\ 2 - \frac{3e^t}{4} + \frac{3te^t}{2} \end{bmatrix}$$

8. (6 points) Find the singular points for $\overset{p}{(x-3)^3 x^2}\frac{d^2y}{dx^2} + \overset{q}{x(x-3)}\frac{dy}{dx} + \overset{r}{(x-3)}y = 0$, and then classify them as regular or irregular. Defend answers w/ work! ✓

Singular points: $\boxed{x = 3, 0}$

$$\lim_{x \to 3} \frac{(x-3) \cdot x(x-3)}{(x-3)^3} \text{ DNE} \Rightarrow x = 3 \text{ is irregular singular}$$

$$\left.\begin{matrix} \lim_{x \to 0} \frac{x \cdot x(x-3)}{(x-3)^3 x^2} = \frac{1}{9} \\ \lim_{x \to 0} \frac{x^2(x-3)}{(x-3)^3 x^2} = \frac{1}{9} \end{matrix}\right\} \Rightarrow x = 0 \text{ is regular singular}$$

9. (9 points) (a) Compute e^{At} if $A = \begin{bmatrix} -1 & -3 \\ 3 & 5 \end{bmatrix}$.

(b) Write the solution for $\frac{d\vec{u}}{dt} = A\vec{u}$ if $\vec{u}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

a) $\tilde{\tau} = 4$, $\Delta = 4 \Rightarrow \lambda = 2, 2$ $\quad \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right] \Rightarrow N(A-2I) = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defect!

$$\therefore e^{At} = e^{2t}\left[I + \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} t\right] = \boxed{e^{2t}\begin{bmatrix} 1-3t & -3t \\ 3t & 1+3t \end{bmatrix}}$$

b) $$\vec{u} = e^{2t}\begin{pmatrix} 1-3t & -3t \\ 3t & 1+3t \end{pmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \boxed{e^{2t}\begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix}}$$

10. (9 points) Solve $\frac{d\vec{u}}{dt} = A\vec{u}$ if $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \cancel{0}\,2 \\ 0 & 0 & 2 \end{bmatrix}$. ω ⊛ ✓

$\lambda = 1, 1, 2$; $\quad \left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow N(A-2I) = c\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

$$\left(A - I \middle| \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow N(A-I) = c\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$ defect!

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \vec{v}^* \in \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$ (Notice $v^* \in \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $k \in \mathbb{R}$ ... I let $k=0$)

$$\boxed{\therefore \vec{u}(t) = c_1 e^{2t}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right)}$$

(Also notice $\vec{v}^*$ is different if you express $N(A-I) = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
Then $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \vec{v}^* \in \begin{pmatrix} k \\ \frac{1}{2} \\ 0 \end{pmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ....)

11. (6 points) A second order equation of the form $\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0$, where $q(x)$ and $r(x)$ are anyalytic functions, uses the power series method to produce the recursion equation $c_n = \frac{c_{n-1} + c_{n-2}}{2n+1}$ for $n \geq 2$. Write the general solution to the equation using the first three nonzero terms for each series solution.

$$\left.\begin{matrix} C_0 = 1 \\ C_1 = 0 \end{matrix}\right\} \Rightarrow C_2 = \frac{1+0}{5} = \frac{1}{5} \; ; \; C_3 = \frac{\frac{1}{5}}{7} = \frac{1}{35} .$$

$$\left.\begin{matrix} C_0 = 0 \\ C_1 = 1 \end{matrix}\right\} \Rightarrow C_2 = \frac{1}{5} \; ; \; C_3 = \frac{1+\frac{1}{5}}{7} = \frac{6}{35} .$$

$$\boxed{\therefore y = A\left(1 + \frac{x^2}{5} + \frac{x^3}{35} + \dots\right) + B\left(x + \frac{x^2}{5} + \frac{6x^3}{35} + \dots\right)}$$

(k) 12. (10 points) Use the Frobenius method to ~~F~~find one solution to $x\frac{d^2y}{dx^2} + y = 0$. Use the first three nonzero terms of a series solution.

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r} = 0 \quad , \quad \underline{C_0 \neq 0} .$$

$$\underset{k=n-1}{} \qquad\qquad \underset{k=n}{}$$

$$\Rightarrow \sum_{k=-1}^{\infty} C_{k+1} (k+r+1)(k+r) x^k + \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$k=-1 \Rightarrow C_0 (r)(r-1) = 0 \Rightarrow \underline{r = 0, 1} ,$$

$$k \geq 0 \Rightarrow C_{k+1} = \frac{-C_k}{(k+r+1)(k+r)} \underset{n=k+1}{\Rightarrow} \boxed{C_n = \frac{-C_{n-1}}{(n+r)(n+r-1)} , \quad n \geq 1}$$

$$r = 0 \Rightarrow C_n = \frac{-C_{n-1}}{n(n-1)} \quad n \geq 1 \; ; \text{ not possible if } n = 1.$$

$$\boxed{\therefore y_1 = x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \dots\right)}$$

$$r = 1 \Rightarrow C_n = \frac{-C_{n-1}}{(n+1)n} .$$

$$C_0 = 1 \Rightarrow C_1 = \frac{-1}{2} \Rightarrow C_2 = \frac{-\left(-\frac{1}{2}\right)}{3 \cdot 2} = +\frac{1}{12} .$$