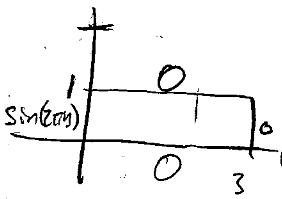
Test 3 100 points Math 220 Name: ______________________

1. (10 points) Solve $u_{xx} + u_{yy} = 0$ if $u(x,0) = 0$, $u(x,1) = 0$, $u(0,y) = \sin(2\pi y)$, and $u(3,y) = 0$. Show the steps discussed in class.



(1) $u = XY \Rightarrow Y'' + \lambda Y = 0;\ Y(0) = 0 = Y(1);\ X'' - \lambda X = 0,\ X(3) = 0$

(2) $\lambda = n^2\pi^2,\ n = 1, 2, \cdots;\ Y_n = \sin(n\pi y);\ X_n = \sinh(n\pi(3-x))$

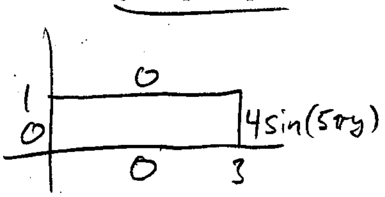
(3) $u(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi(3-x)) \sin(n\pi y)$

(4) $\sin(2\pi y) = u(0,y) = \sum_{n=1}^{\infty} c_n \sinh(3n\pi) \sin(n\pi y) \Rightarrow 1 = c_2 \sinh(6\pi),\ c_n = 0$ otherwise

$$\Rightarrow \boxed{u(x,y) = \frac{1}{\sinh(6\pi)} \sinh(2\pi(3-x)) \sin(2\pi y)}$$

2. (6 points) Solve $u_{xx} + u_{yy} = 0$ if $u(x,0) = 0$, $u(x,1) = 0$, $u(0,y) = 0$, and $u(3,y) = 4\sin(5\pi y)$. You may skip steps.

$X(0) = 0$



$\lambda = n^2\pi^2,\ n = 1, 2, \cdots;$

$Y_n = \sin(n\pi y);\ X_n = \sinh(n\pi x)$

$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x) \sin(n\pi y)$

$4\sin(5\pi y) = u(3,y) \Rightarrow c_5 = \frac{4}{\sinh(15\pi)} \Rightarrow \boxed{u(x,y) = \frac{4\sinh(5\pi x)\sin(5\pi y)}{\sinh(15\pi)}}$

3. (2 points) Write the solution for $u_{xx} + u_{yy} = 0$ if $u(x,0) = 0$, $u(x,1) = 0$, $u(0,y) = \sin(2\pi y)$, and $u(3,y) = 4\sin(5\pi y)$. You do not need to show any work.

Superposition $\Rightarrow \boxed{u(x,y) = \frac{\sinh(2\pi(3-x))\sin(2\pi y)}{\sinh(6\pi)} + \frac{4\sinh(5\pi x)\sin(5\pi y)}{\sinh(15\pi)}}$

T3, 2

4. (9 points) Solve $y''(t) + 6y'(t) + 8y = (t^2 - 1)\delta(t-2)$ if $y(0) = 2$ and $y'(0) = 1$.

$$\xrightarrow{\mathcal{L}} s^2 Y - 2s - 1 + 6sY - 12 + 8Y = 3e^{-2s}$$

$$\Rightarrow (s^2+6s+8)Y = 2s + 13 + 3e^{-2s}$$

$$\Rightarrow Y = \frac{2s+13}{(s+4)(s+2)} + e^{-2s}\left(\frac{3}{(s+4)(s+2)}\right)$$

$$\Rightarrow Y = \frac{-\frac{5}{2}}{s+4} + \frac{\frac{9}{2}}{s+2} + e^{-2s}\left(\frac{-\frac{3}{2}}{s+4} + \frac{\frac{3}{2}}{s+2}\right)$$

$$\Rightarrow \boxed{y(t) = -\frac{5}{2}e^{-4t} + \frac{9}{2}e^{-2t} + \frac{3}{2}H(t-2)\left(e^{-2(t-2)} - e^{-4(t-2)}\right)}$$

OR $Y = \frac{2(s+3)+7}{(s+3)^2-1} + e^{-2s}\left(\frac{3}{(s+3)^2-1}\right)$

$$\Rightarrow \boxed{y(t) = 2e^{-3t}\cosh(t) + 7e^{-3t}\sinh(t) + 3H(t-2)e^{-3(t-2)}\sinh(t-2)}$$

(*)

$t * y$

5. (8 points) Solve $y(t) = H(t-2) \cancel{+} - \int_0^t (t-\tau)y(\tau)\,d\tau$.

$$\xrightarrow{\mathcal{L}} Y = \frac{e^{-2s}}{s} - \frac{1}{s^2}Y$$

$$\Rightarrow Y\left(1 + \frac{1}{s^2}\right) = \frac{e^{-2s}}{s}$$

$$\Rightarrow Y = \frac{e^{-2s}}{s} \cdot \frac{1}{\left(1+\frac{1}{s^2}\right)}$$

$$\Rightarrow Y = e^{-2s} \cdot \frac{1}{s + \frac{1}{s}} = e^{-2s} \cdot \frac{s}{s^2+1}$$

$$\Rightarrow \boxed{y(t) = H(t-2)\cos(t-2)}$$

T3, 3

6. (9 points) Find the eigenvalues and eigenfunctions for $x''(t) + \lambda x(t) = 0$, $x(0) = 0$, and $x'(\pi) = 0$. Show detailed work.

$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

$$\Rightarrow x(t) = \begin{cases} A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t), & \lambda > 0 \\ A + Bt, & \lambda = 0 \\ A\cosh(\sqrt{-\lambda}t) + B\sinh(\sqrt{-\lambda}t), & \lambda < 0 \end{cases}$$

$$0 = x(0) = \begin{cases} A \\ A \\ A \end{cases} \Rightarrow A = 0 \ \forall \lambda.$$

$$0 = x'(\pi) = \left.\begin{cases} B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi), & \lambda > 0 \\ B, & \lambda = 0 \\ B\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\pi), & \lambda < 0 \end{cases}\right\} \Rightarrow B = 0$$

$$\Rightarrow \sqrt{\lambda}\pi = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots$$

$$\boxed{\therefore \lambda = \frac{(2n+1)^2}{4}, \ n = 0, 1, 2, \ldots \quad \text{and} \quad X_n = B\sin\left(\frac{(2n+1)t}{2}\right)}$$

are the eigenvalues and eigenfunctions.

7. (8 points) Solve $u_t = u_{xx}$ if $u(0,t) = 0$, $u(2,t) = 0$, and $u_t(x,0) = x$. Show the steps discussed in class.

(1) $u = XT \Rightarrow X'' + \lambda X = 0$, $X(0) = 0 = X(2)$; $T' = -\lambda T$

(2) $\lambda = \frac{n^2\pi^2}{4}$, $n = 1, 2, \ldots$; $X_n = \sin\left(\frac{n\pi}{2}x\right)$, $T_n = e^{-\frac{n^2\pi^2}{4}t}$

(3) $$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2}x\right) e^{-\frac{n^2\pi^2}{4}t}$$

(4) $$x = u_t(x,0) = \sum_{n=1}^{\infty} C_n\left(\frac{-n^2\pi^2}{4}\right)\sin\left(\frac{n\pi}{2}x\right).$$

	x	$\sin\left(\frac{n\pi x}{2}\right)$
$+$	x	$\sin\left(\frac{n\pi x}{2}\right)$
$-$	1	$-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}$
	0	$-\sin\left(\frac{n\pi x}{2}\right)\cdot\frac{4}{n^2\pi^2}$
		0

F.S. X_{ODD}: $a_n = 0$; $b_n = \frac{2}{2}\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$

$$\Rightarrow C_n\left(\frac{-n^2\pi^2}{4}\right) = -\frac{4}{n\pi}(-1)^n.$$

$$\boxed{\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{16}{n^3\pi^3}(-1)^n \sin\left(\frac{n\pi}{2}x\right) e^{-\frac{n^2\pi^2}{4}t}}$$

T3, 4

8. (10 points) Solve $y_{tt} = 4y_{xx}$ if $y_x(0,t) = 0$, $y_x(1,t) = 0$, $y(x,0) = 1 - 2\cos(\pi x)$, and $y_t(x,0) = 0$. Show the steps discussed in class.

① $y = XT \Rightarrow X'' + \lambda X = 0,\ X'(0) = 0 = X'(1);\ T'' + 4\lambda T = 0,\ T'(0) = 0.$

② $\lambda = n^2\pi^2,\ n = 0,1,2,\dots;\ X_n = \cos(n\pi x),\ (X_0 = 1);\ T_n = \cos(2n\pi t)$
$(T_0 = 1)$

③ $y(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2n\pi t)\cos(n\pi x)$

④ $1 - 2\cos(\pi x) = y(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\pi x)$

$\Rightarrow C_0 = 1,\ C_1 = -2.$

$$\boxed{\therefore\ y(x,t) = 1 - 2\cos(2\pi t)\cos(\pi x)}$$

9. (6 points) Use past work to help solve $y_{tt} = 4y_{xx}$ if $y_x(0,t) = 0$, $y_x(1,t) = 0$, $y(x,0) = 0$, and $y_t(x,0) = 1 - 2\cos(\pi x)$.

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$T'' + 4\lambda T = 0,\ T(0) = 0.$

(#8) $\Rightarrow \lambda = n^2\pi^2,\ n = 0,1,2,\dots;\ X_n = \cos(n\pi x),\ X_0 = 1;\ T_n = \sin(2n\pi t),\ T_0 = t.$

$\therefore\ y(x,t) = C_0 t + \sum_{n=1}^{\infty} C_n \sin(2n\pi t)\cos(n\pi x)$

and $1 - 2\cos(\pi x) = y_t(x,0) = C_0 + \sum_{n=1}^{\infty} 2n\pi C_n \overset{1}{\cancel{\cos(0)}}\cos(n\pi x)$

$\Rightarrow C_0 = 1,\ -2 = 2\pi C_1 \Rightarrow C_1 = -\frac{1}{\pi}.$

$$\boxed{\therefore\ y(x,t) = t - \frac{1}{\pi}\sin(2\pi t)\cos(\pi x)}$$

T3, 5

10. (8 points) Use the definition of Laplace transform to show that $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. ~~You may assume the derivatives of $f(x)$ exist for all orders.~~

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-st}\,dt$$

$\oplus\ e^{-st}$	$f'(t)$
$\ominus\ -se^{-st}$	$f(t)$

$$= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= [0 - f(0)] + s\,\mathcal{L}(f(t))$$

$$= s\,\mathcal{L}(f(t)) - f(0) \quad \boxtimes$$

11. (8 points) Find $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2(s^2+1)}\right\}$. $= f(t)$

$$\frac{s-1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

$$A = \frac{s-1}{s^2+1}\Big|_{s=0} = -1$$

$$B = \frac{d\,\frac{s-1}{s^2+1}}{ds}\Big|_{s=0} = \frac{(s^2+1)-(s-1)(2s)}{(s^2+1)^2}\Big|_{s=0} = 1$$

$$C(i)+D = \frac{s-1}{s^2}\Big|_{s=i} = \frac{i-1}{-1} \Rightarrow C=-1,\ D=1$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s-1}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(-\frac{1}{s^2} + \frac{1}{s} + \frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$$

$$\therefore f(t) = \boxed{-t + 1 + \sin(t) - \cos(t)}$$

OR

$$\frac{s-1}{s^2(s^2+1)} = \frac{\cancel{s}\,1}{s\cancel{s}(s^2+1)} - \frac{1}{s^2(s^2+1)}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} - \frac{1}{s^2} + \frac{1}{s^2+1}$$

$\Rightarrow f(t)$

$$= 1 - \cos(t) - t + \sin(t)$$

T3,6

12. (9 points) Let $f(x) = H(x-\pi)$ for $0 \le x \le 2\pi$. Find the first three nonzero terms of the cosine series expansion for $f(x)$.

F.S. f_{even} : $b_n = 0$ $\qquad \lambda = \frac{n^2\pi^2}{(2\pi)^2} = \frac{n^2}{4} \Rightarrow \sqrt{\lambda} = \frac{n}{2}$.

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} H(x-\pi)\cdot 1\,dx = \frac{1}{\pi}\int_\pi^{2\pi} dx = 1 \Rightarrow \frac{a_0}{2} = \frac{1}{2}.$$

$$a_n = \frac{2}{2\pi}\int_\pi^{2\pi} \cos\left(\frac{n}{2}x\right)dx = \frac{1}{\pi}\sin\left(\frac{n}{2}x\right)\cdot\frac{2}{n}\Big|_\pi^{2\pi} = \frac{2}{n\pi}\left[\cancelto{0}{\sin(n\pi)} - \sin\left(\frac{n\pi}{2}\right)\right].$$

$$\boxed{\therefore \text{F.S. } f_{Even} = \frac{1}{2} - \frac{2}{\pi}\cos\left(\frac{x}{2}\right) + \frac{2}{3\pi}\cos\left(\frac{3x}{2}\right) - \dots}$$

13. (9 points) Solve $u_t = u_{xx}$ if $u_x(1,t) = 0$, $u_x(2,t) = 0$, and $u(x,0) = \cos(\pi x)$. Show the steps discussed in class.

① $u = XT \Rightarrow X'' + \lambda X = 0,\ X'(1) = 0 = X'(2);\ T' = -\lambda T$

② $\lambda = \frac{n^2\pi^2}{1^2},\ n = 0,1,2,\dots;\ X_n = \cos(n\pi(x-1));\ T_n = e^{-n^2\pi^2 t}$

③ $u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t}\cos(n\pi(x-1))$

Note: if you completely ignore the shift, you will still be correct since $\cos(n\pi(x-1))$ and $\cos(n\pi x)$ are in the same espace.

④ $\cos(\pi x) = u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\pi(x-1))$

Since $\cos(\pi(x-1)) = \cos(\pi x - \pi) = -\cos(\pi x)$;

$C_1 = -1$.

$\therefore \boxed{u(x,t) = -e^{-\pi^2 t}\cos(\pi(x-1))}$ or $\boxed{e^{-\pi^2 t}\cos(\pi x)}$.